Chapter Eighteen

Developing Topsy Turvy and Number Planet

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18.1 Summary

This article is about a mathematics collaboration in the spirit of Martin Gardner. Igor Kriz illustrated group theory and the sporadic simple Mathieu group $M12$ [1] in *Scientific American* [2] by turning them into permutation puzzles. He challenged readers to find a mechanical implementation of his $M12$ puzzle.

Oskar van Deventer took up the challenge. The resulting two completely different implementations, *Topsy Turvy* and *Number Planet*, are described in this article.

18.2 Challenge in Scientific American

The July 2008 issue of Scientific American featured an article by Igor Kriz on group theory [2]. The purpose of the article was to educate people on group theory. It explained about *simple groups*, which are the group-theory equivalent of prime numbers. It highlighted researchers' multidecade mathematical quest to identify and classify all simple groups. And it illustrated the concept of a *simple sporadic group* with a set of electronic puzzles, programmed by Igor's graduate student Paul Siegel. One of the puzzles was the *M12* puzzle, based on the simple sporadic Mathieu 12 group.

The *M12* puzzle is played as follows. Take twelve tokens, numbered 1 to 12. There are two permutations: *Invert* and *Merge* (see Figure 8.1). The object of the *M12* puzzle is similar to that of the Rubik's Cube: to unscramble it by using only the two permutations.

$$\text{Invert} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\text{Merge} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 3 & 5 & 7 & 9 & 11 & 12 & 10 & 8 & 6 & 4 & 2 \end{pmatrix}$$

Figure 18.1: The *M12* puzzle uses two permutations called *Invert* and *Merge*.

Igor suggested in the article that the three puzzles might be mechanically implemented, perhaps using a rotating device and a system of gears. He left it as a challenge to the reader.

18.3 Implementation #1: *Topsy Turvy*

Oskar van Deventer took up the challenge to implement the puzzle using a set of tokens numbered 1 through 12. The *Invert* operation is easily implemented, as one can just turn around the set of tokens using some sort of mechanical device. The *Merge* operation is more challenging. Oskar's first idea was to implement the *Merge* using some slapstick construction, but he could not find a good mechanism. However, the reverse operation to *Merge*, which he christened *Split*, might be easier to implement. Oskar contacted Igor, who confirmed that *Split* could be used.

Oskar had previously developed several puzzle mechanisms to manipulate dropping tokens called *Jukebox* and *Pachinko* (see Figure 18.2. The former uses physical switches to alternate moving tokens left and right. The latter uses a pattern of grooves that holds one or two tokens until one more token is inserted that pushes the other tokens down. Oskar found that the latter mechanism could be used to build the 12-splitters needed for the *Split* operation.

Figure 18.3 shows a working prototype of *Topsy Turvy*, which implements the 12-splitter. A big crank is used to move the 12 tokens into the 12-splitter. The *Invert* operation is implicitly implemented, as the crank can be turned either to the left or right. When turned, the 12 tokens are dropped in one by one. The first 11 of the tokens will then land stably on top of one another. However, when the final 12th token drops, it rolls down over the 11th, and while dropping, it pushes the 11th out of position. Then the 11th pushes the 10th out of position, which pushes the 9th, which pushes the 8th, an so on. In this cascade, the whole stack of tokens falls apart, with the even tokens moving to the right and the odd ones to the left.

Oskar had to build several prototypes to get the mechanism

Figure 18.2: The *Jukebox* and *Pachinko* puzzle mechanisms designed by Oskar

Figure 18.3: *Topsy Turvy*, featuring a 12-splitter.

right. The prototypes were built using the laser cutter of Peter Knoppers, using MDF and acrylic. The tokens are made of cast tin, using laser-cut MDF moulds. In order to have the mechanism turn smoothly, four large gears act as a ball bearing, carrying the weight of the crank mechanism. A rattle mechanism is used to force a user to finish a move once started, preventing illegal moves.

The first prototype had the major flaw that a user could continue turning while the tokens were still dropping. Allowing that to happen causes tokens to collide when they are caught by the crank mechanism at the bottom, enabling both illegal moves and blockage of the intended mechanism. George Miller found an elegant solution to this problem: a toggle switch which limits the rotation of the crank between -240 and +240 degrees. With the switch in place, a user has to turn the crank all the way back, which takes sufficient time for the tokens to settle at the bottom. Magnets were used to make the switch bi-stable.

Another problem was that tokens could skip the entry if the crank is turned too fast. Peter Knopper's solution was to place a pin at the top entry of the grooves, which forces the tokens down.

With all major problems solved, the third prototype was found to work in a satisfactory way.

Figure 18.4: The gears, rattle mechanism, and toggle switch used to prevent illegal moves in *Topsy Turvy*.

18.4 Implementation #1: *Number Planet*

While Oskar was working on *Topsy Turvy*, Igor suggested a completely different implementation. Igor had found special *planar permutations* that also implement the *M12* group. Two permutations are called *Rotate* and *Swap*. See Figure 18.5.

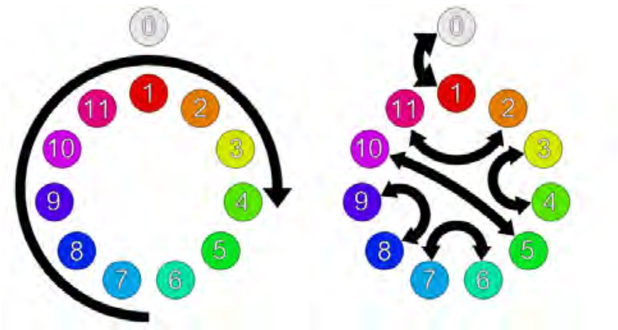

Figure 18.5: The *Rotate* and *Swap* permutations

Oskar started 3D sketching. After a lot of communication with Igor, a mechanism was found that could do the trick. See Figure 18.6.

However, Oskar was not satisfied with the mechanism and its round tokens. A much better mechanism might be possible if the 0 and 1 were not surrounded by the 11-2 swap. Oskar asked Igor whether there might not exist a better planar *M12* permutation. Igor started looking using a *Maple* program, while Oskar used a Python program written by George Miller. With crossing emails, Igor beat Oskar by only five minutes, both discovering that the requested planar permutation does indeed exist.

Using this permutation Oskar made a 3D design that used trapezoid-shaped tokens that push each other better when performing the *Rotate* operation. See Figure 18.7.

The resulting puzzle called *Number Planet* by Igor, was proto-

Figure 18.6: Mechanism implementing *Rotate* and *Swap*.

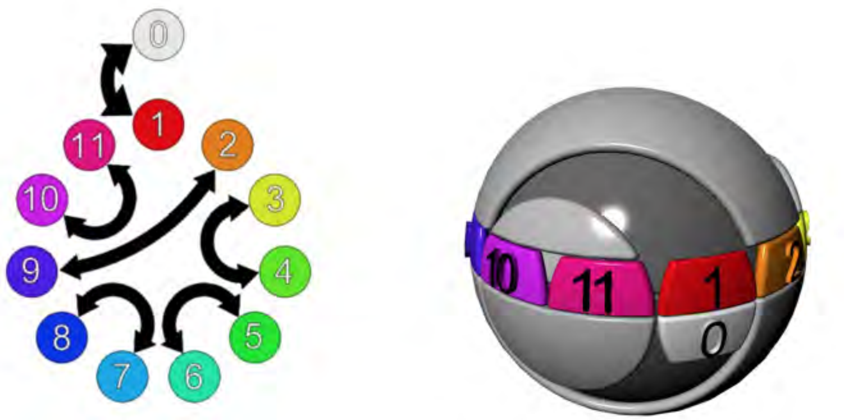

Figure 18.7: A better planar permutation enabling trapezoid-shaped tokens.

typed using 3D printing technologies. The first prototype, made by UM3D in ABS-based fusion deposition modeling, was a failure because Oskar had made a modeling error. A second prototype owked reasonably well, but it was still cumbersome to assemble with it's many subassemblies.

A third prototype made by TNO Netherlands using nylon-based selective laser sintering and painted with textile acid dye, was finally a success.

18.5 Solving the *M12* puzzles

With working prototypes available, the reader may wonder how these puzzles should be solved. First of all, it should be noted that both implementations feature two permutations, albeit different ones.

Figure 18.8: The *Number Planet* design, and FDM and SLS prototypes

For *Topsy Turvy*:

Left: 1-2-3-4-5-6-7-8-9-10-11-12 $\rightarrow$ 11-9-7-5-3-1-2-4-6-8-10-12.

Right: 1-2-3-4-5-6-7-8-9-10-11-12 $\rightarrow$ 2-4-6-8-10-12-11-9-7-5-3-1.

For *Nuumber Planet*:

Rotate: 0-1-2-3-4-5-6-7-8-9-10-11 $\rightarrow$ 0-2-3-4-5-6-7-8-9-10-11-1.

Swap: 0-1-2-3-4-5-6-7-8-9-10-11 $\rightarrow$ 0-1-9-4-3-6-5-8-7-2-11-10.

Secondly, both mechanisms implement the *M12* group, which has

$$12 \times 11 \times 10 \times 9 \times 8 = 95040$$

permutations. The *M12* group has the property that if five token are at their correct place, then the other seven tokens are correct too [1]. Using this information one could envision the following solution approaches.

- **God's Table by computer**: As the solution space is very small, it is quite feasible to build a "GodâĂŹs Table" enumerating all possible states and the solution sequence for each state. This is exactly what George Miller did. The file that his Python program produced is small enough to be used on a smart phone, so you can always have a solution at hand. Although the God's Table provides the fastest solution, it is impossible for a human being to memorize.
- **Recursive solution by hand**: A solution worked out by Igor uses a small set of recursive operations. While much easier to memorize, the recursiveness of the solution requires many (thousands?) of moves. This makes the solution rather impractical for the mechanical versions.
- **Computer-aided optimization**: A computer could be used to find the shortest five sets of operations that brings 5 tokens

into their correct places one by one, using the property of *M12* mentioned above.

18.6 Mission accomplished; now what?

This article presented two completely different implementations for Igor KrizâĂŹ *M12* puzzle challenge. *Topsy Turvy* uses gravity and its operations are non-reversible. *Number Planet* is a twisty puzzle and every operation can be undone. *Topsy Turvy* occasionally cascades too early if it is handled too wildly. The prototypes are not perfect. The rough surfaces of the 3D-printed Number Planet prototypes are sometimes a bit sticky.

Both puzzles are excellent illustrations of *M12* simple sporadic group. They would make interesting collector's items for connoisseurs. At the moment of writing this article, it is unclear whether the puzzles have any further commercial potential.

Further Reading

Kriz, I. & Siegel, P. (2008). Simple groups at play. *Scientific American*, *299*(1), 84–89. doi:10.1038/scientificamerican0708-84

Wikipedia. (2015). Mathieu group m12 — wikipedia, the free encyclopedia. [Online; accessed 25-September-2016]. Retrieved from https://en.wikipedia.org/w/index.php?title=Mathieu_group_M12&oldid=697630590