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COMPUTER RECREATIONS

Old and new three-dimensional mazes

by A. K. Dewdney

"...a labyrinth constructed by Daedalus, so artfully contrived that whoever was enclosed in it could by no means find his way out unassisted."
—*Bulfinch's Mythology*

Most mazes are two-dimensional, so that if we look down on them, we are able to work our way through their intricate twists and turns. But we cannot look down, so to speak, on three-dimensional mazes: upper levels obscure lower ones. We have no alternative but to feel our way—either literally or figuratively—along complex passages.

There are both old three-dimensional mazes and new ones. Given its legendary difficulty, Daedalus' labyrinth of yore must have been three-dimensional. Its stygian darkness serves as an appropriate setting for an exploration of maze-solving techniques, including an extension of the famous right-hand rule employed in the solution of two-dimensional mazes. As for modern mazes, those that are constructed by M. Oskar van Deventer are not only three-dimensional but also invisible! They lead to a fascinating reconstruction problem: When can three two-dimensional mazes define a single three-dimensional one?

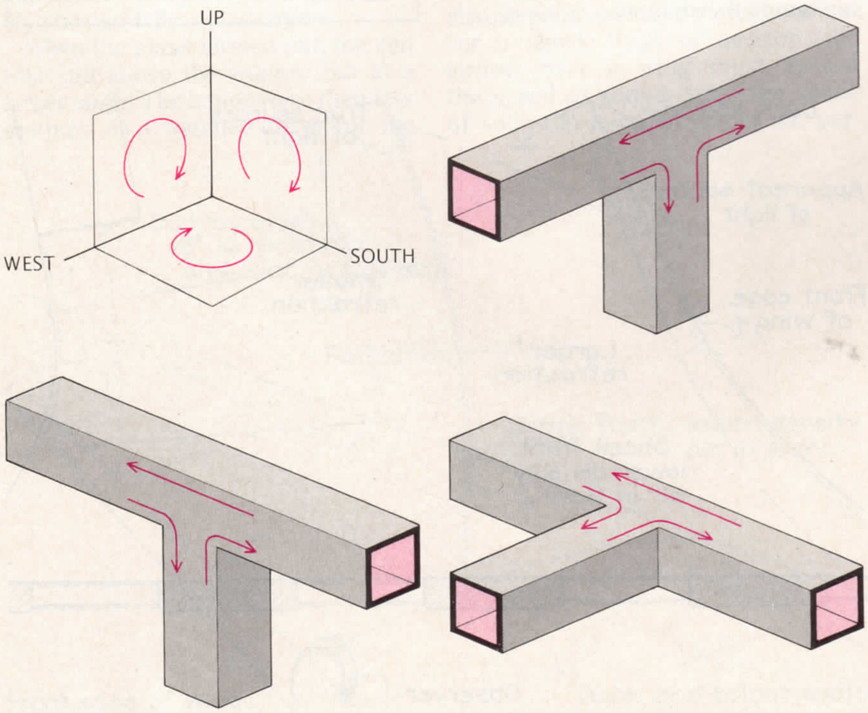
Daedalus constructed his notorious labyrinth for Minos, the powerful king of Crete. The king did not intend the maze for recreational purposes, however. The maze served to confine seven youths and seven maidens sent to him by Athens each year as tribute. No amount of intelligence would serve the hapless victims as they crept along its damp, dark passages seeking the way out. But that was not the worst of it: a fierce and horrendous creature, known as the Minotaur, inhabited the labyrinth. The Minotaur, which had a human head and the body of a bull, devoured the poor young Athenians who were trapped in the structure.

Only Theseus, the fabled Greek hero, solved the labyrinth and in doing so killed the Minotaur. He tied a length of thread provided by the king's daughter (who secretly loved him, of course) to the outside of the labyrinth and unwound the thread as he wandered the passages searching for the Minotaur. After slaying the Minotaur, he merely followed the thread back to escape from the labyrinth.

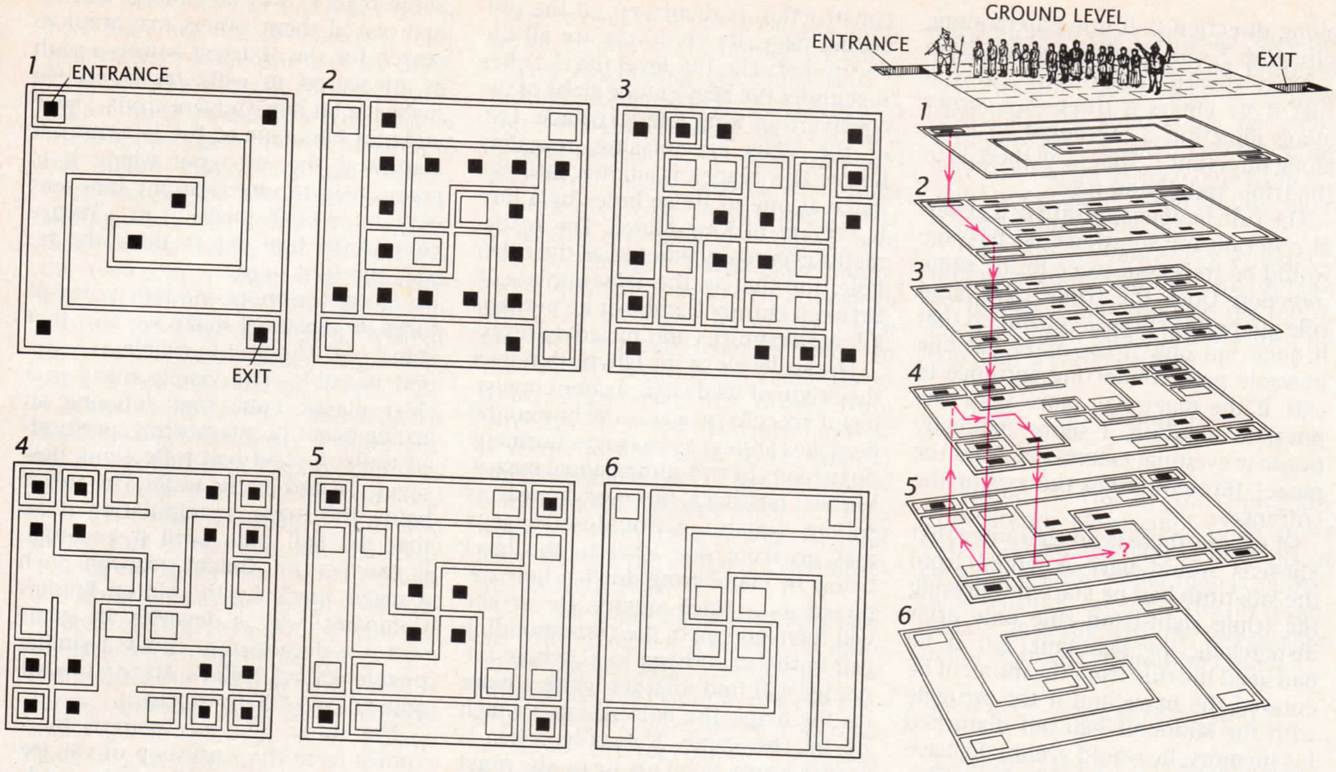
What would have happened if Theseus had been absentminded and had forgotten to secure the thread before entering the labyrinth? Could he have escaped some other way? One possibility would have been for him to tie the thread to the slain Minotaur before setting out in search of the exit. The thread would then at least have enabled him to return to the same starting point (the carcass of the monster) after each unsuccessful probe into Daedalus' cunning labyrinth. But the thread alone would not have guaranteed an eventual exit from the maze. How would Theseus have remembered which passages he had already explored?

That depends on whether Theseus' memory was internal or external. If it was internal, he might simply have remembered the turns he took at each junction he encountered. If it was external, he might have placed some token at each junction as a record of transversal. Personally, I think the latter form of memory is the likelier one, and so we shall allow Theseus a number of one-drachma coins.

Every time Theseus came to a junction of three or more passages, he would have examined the floor at the entrance to each passage. If a coin was already there, he would not have entered that passage. If no coin was there, he would have entered. After entering he would immediately have put a coin down within view of the junction. Some readers will no doubt object that Theseus could not possibly have seen the coins because of the utter darkness in the labyrinth. It therefore seems reasonable to allow



The triple right-hand rule for three-dimensional mazes



The reconstruction of Daedalus' labyrinth and a possible path **(color)**

him some form of ancient fire-making apparatus, such as a cigarette lighter.

In following the procedure just outlined, Theseus might well have been forced to backtrack. If, for example, he had come to a dead end or a junction where all passages had coins at their entrances, he would have had to retrace his steps. Is it possible that in the course of backtracking Theseus might have encountered nothing but nonenterable junctions? In other words, could he have been caught in an infinite loop? For the benefit of those who like to think for themselves, I shall not answer the question. Suffice it to say that the basic method I have outlined is widely applied in modern computing for searching through data structures; it is called a depth-first search.

It might have happened that our intrepid hero had no coins or—worse yet—no cigarette lighter. How could he then have solved the maze? Luckily there is a method for escaping from the maze without external memory. Moreover, executing it would not have taxed Theseus' brain any more than carrying out a depth-first search. I call this method the triple right-hand rule.

Ordinarily two-dimensional mazes can be solved by the so-called right-hand rule: on entering the maze one keeps a wall continuously on one's right, no matter how the passages may twist and turn. If a passage forks, one turns down the right-hand corridor. If a passage comes to a dead end, one simply turns around—keeping a wall on one's right. Eventually one will emerge. Astute readers will have noticed I did not state explicitly that one would emerge at the exit. If the exit happens to be in the middle of the maze (as it is in many paper-and-pencil exercises), one might well come out where one entered after a tiring application of the right-hand rule. But reemerge one must. The reason is very simple: abiding by the rule enables one never to retrace one's steps. If no part of the wall that defines the maze's passages is ever retraced, it follows that eventually one must run out of wall, so to speak, at an opening. (The exception is the case in which the walls of the maze form a completely closed circuit, but in that case there would be no opening in the wall where one could have entered.)

A variation of the right-hand rule can be applied in solving three-dimensional mazes, including the cruel labyrinth of King Minos. To make things simple, I assume all passages in the maze have a square cross section and are quite straight, except at bends where they make 90-degree turns. In addition, I assume the passages run precisely east-west, north-south or up-down and are therefore perpendicular to one another. I also assume only two types of junction are formed wherever three passages come together: a T junction and a three-way corner.

Let me now take the reader groping along the passages in order to explain the operation of the triple right-hand rule. No gyroscope is necessary; gravity tells us which way is up and which is down. The other four directions are remembered by keeping track of our turns as we make our way through the three-dimensional maze. If we enter the maze facing east, for example, a turn to the right leaves us facing south; after another turn to the right we would be facing west, and so on.

The triple right-hand rule is applied at a T junction only after we identify the plane in which the junction lies, since each of the three possible planes has a specific "handedness" assigned to it [*see illustration on opposite page*]. Imagine a clock stuck on a surface parallel to the plane of the T junction. We arbitrarily call the direction in which the hands of a clock turn "right," and we will consistently turn in that direction in the labyrinth.

In the case of three-way corners the rule must be modified somewhat. (Is that the Minotaur bellowing in the distance?) Say that up-down passages have direction 1, north-south passages direction 2 and east-west passages direction 3. If one enters a three-way corner along direction 1, one leaves along direction 2. If one enters along direction 2, one leaves along direction 3. Even Theseus might have guessed that if he enters a three-way corner along direction 3, he ought to leave along direction 1. That is all there is to the triple right-hand rule.

The rule happens to satisfy a general criterion for solutions: no passage would be traversed twice in the same *direction.* Does the triple right-hand rule guarantee success? I contend that it does, but only if the maze has one possible path connecting entrance to exit. If the maze has more than one possible solution, I more modestly propose eventual emergence from the maze—through either the exit or the entrance.

Of course, there is no guarantee that Theseus would have emerged from the labyrinth had he started following the triple right-hand rule only after dispatching the Minotaur. But if he had used the rule from the moment he entered the maze and if the struggle with the Minotaur had not disturbed his memory, he would eventually have emerged a hero into the daylight. Theseus would not have cared whether he had left by the entrance or the exit!

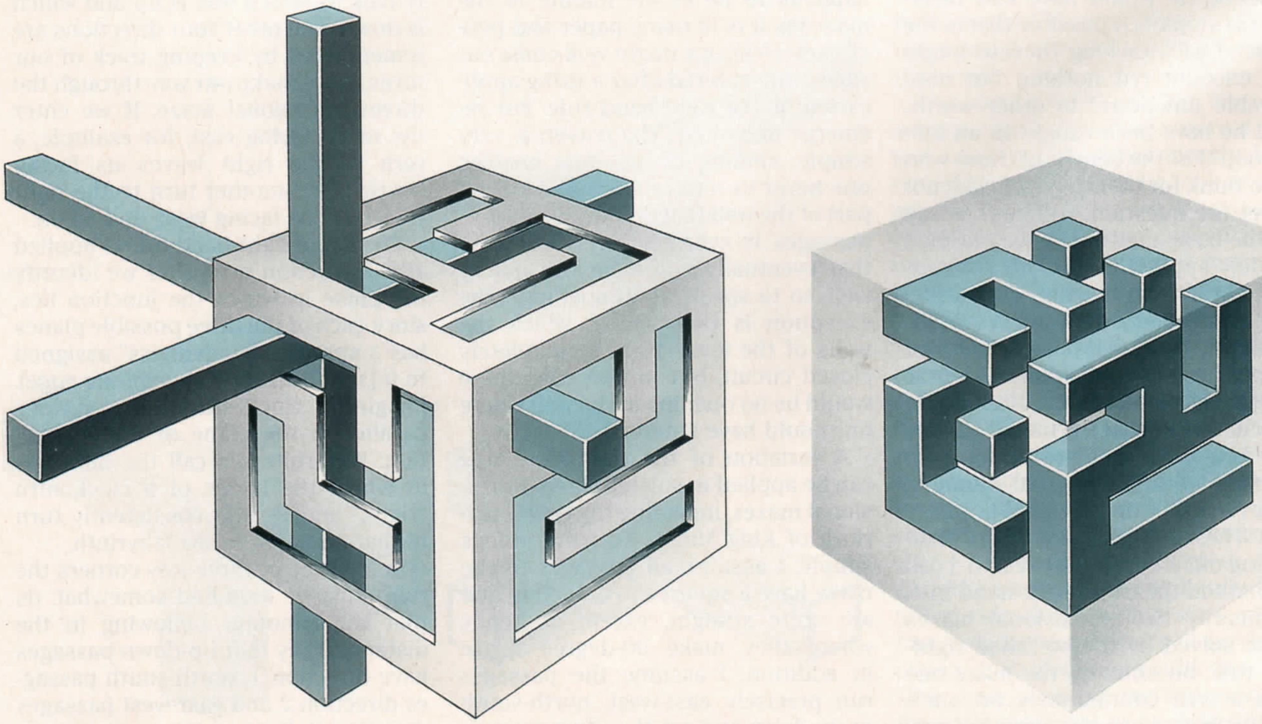
With these rules in mind readers may feel ready to try solving a three-dimensional labyrinth. After some extensive research into the matter I offer nothing less than a reconstruction of Daedalus' original labyrinth. The reconstruction is displayed on the preceding page. Its six levels are all underground. The top level (level 1) lies just under the heavy stone slabs of the courtyard of King Minos' palace. Two of the slabs are missing, revealing holes. The reader is shown into the maze at one of those holes by a burly servant of King Minos. The reader might eventually emerge at the other hole, for that is the labyrinth's exit. Between entrance and exit lie perhaps a few adventures and misadventures.

The six levels of the labyrinth reflect the origin of its design: a cube consisting of six cells on a side. All horizontal passages appear as passages normally do in maps of two-dimensional mazes. Vertical passages, however, appear as solitary squares. A would-be Theseus may go from one level to the level below by clambering down a hole depicted as a black square. He or she will then emerge in the corresponding cell in the underlying level, where the reader will find a larger white square on the map—the hole through which he or she came. Naturally, when a reader wants to go up, he or she must go to the nearest white square. Sometimes one sees a black square inside a white one. This simply means that from that particular position in the labyrinth it is possible to go either up or down.

There are several possible paths from entrance to exit in the labyrinth. Some readers will be content to find just one of them; others may want to search for the shortest solution path as measured in cells traversed. To make it a bit more adventurous, I have added a Minotaur to the labyrinth. It stands at the one spot where it is guaranteed to intercept any innocent explorer. I shall mention in a future column the first five readers who report the spot to me.

Among the more modern types of three-dimensional maze are two that stand out. The first is visible, the second invisible. The visible maze is a clear plastic cube that contains an arrangement of intersecting, perforated walls. A steel ball rolls along passages created by the walls. The solver holds the maze, manipulating it so that the ball rolls until it eventually reaches the "finish" position. Such a maze, made by the Milton Bradley Company, was a favorite in game stores a decade ago. Today a similar puzzle, called Miller's Maze, is available at Toys-"R"-Us stores.

The other kind of modern maze comes from the workshop of van Deventer in Voorburg, The Netherlands. He calls one of his productions a *holle doolhof,* or hollow maze. The terminology is perfectly reasonable: his mazes are wood boxes that contain absolutely nothing! Not a passage or wall can be seen within, but a three-dimensional maze nonetheless exists in the box.



***A simple van Deventer maze* (left) *and its projective cast* (right)**

The secret lies in the sides of the box. They are two-dimensional "control mazes": wood surfaces in which slots have been cut. A cursor consisting of three mutually perpendicular wood spars registers one's position in the hollow maze. Each spar passes from one side of the box to the other, sliding along the slots of the control maze on each side. Not surprisingly, the two control mazes on opposite sides of the box are identical. In this way van Deventer can produce a single three-dimensional maze from three pairs of two-dimensional mazes.

In the simple example shown on the opposite page one starts with the cursor in one corner of the box and tries to manipulate it into the opposite corner. Each spar is pushed into and pulled out of the box, automatically moving the other two spars (if possible) along slots in their respective control mazes. It might seem that to solve a hollow maze one merely solves each of the three control mazes. But this is not so. Although each of the control mazes can be easily solved, the hollow maze is quite difficult.

The difficulty lies in the fact that possible moves on one control maze may be blocked by another control maze. Moreover, it is not clear in which order the cursor's three spars should be pushed or pulled. There may be several possible moves at any given position of the cursor. To solve the maze one might as well close one's eyes and "feel" one's way through it. In such a mode the invisible maze within the box takes on a new, tactile reality.

The invisible maze can in principle be made visible by tracing the slots of three mutually adjacent control mazes onto a solid cube of material that can be carved. When the tracing is complete, all "nonmaze" material is cut away from the solid. The control mazes must of course have the same orientation as they do on the original van Deventer maze.

Since the process is largely imaginary to begin with, I equip myself with a laser saw in carrying it out. Positioning the saw directly over one of the cube's faces, I simply follow the traced lines, cutting straight through the solid as I go. When the cutting is complete, I gingerly push all the unwanted material out of the cube. It slides away, leaving a three-dimensional form that corresponds to the slots of the control maze. After the same process is repeated for the other two faces, the solid that remains is in effect a "negative" of van Deventer's implicit maze: the allowed passages are represented by solid posts and beams. I call it a projective cast. Readers can see a rendition of one on the opposite page.

Two fascinating questions revolve around projective casts. First, when do three two-dimensional mazes yield a projective cast of a viable three-dimensional maze? Second, when does a projective cast yield three projections that are viable two-dimensional mazes? The term "viable maze" ought to be defined. It refers to a maze in which all passages have unit width and there is a path from the "start" to the "finish" position. (A viable three-dimensional maze that results from three two-dimensional control mazes, it seems to me, ought to be called a van Deventer maze.) Other conditions could readily be suggested, but they would relate to the aesthetics of good maze design; the proposed definition will do for a start.

One can experiment with very simple control mazes and still be quite confounded by what emerges in three dimensions. For example, one can construct a van Deventer maze from rather trivial control mazes consisting of 3-by-3 cellular matrices in which certain adjoining cells have been removed. Readers might enjoy starting with three 3-by-3 control mazes having L-shaped slots in various orientations. How many combinations result in van Deventer mazes?

The other question addresses the opposite issue: When does a three-dimensional maze yield three viable control mazes as projections? Both questions have practical importance for van Deventer. An answer to either one greatly simplifies the process of designing his mazes. Van Deventer himself confesses to having done a great deal of actual cutting and trying in coming up with more complicated hollow mazes. Readers with something to say on the subject should write to van Deventer directly at the following address: p.a. Dr. Beguinlaan 44, 2272 AK Voorburg, The Netherlands. A creative response might well result in a second look at van Deventer mazes in this department.

The invisible professor appeared in this department in May to draw a number of classic examples from the infinite variety of trigonometric and algebraic curves. Among the many readers who had made prior acquaintance with the professor were some who had interesting comments to make.

Abe Achkinazi of Bell, Calif., has proposed a date between the invisible professor and Lucy, the Hewlett-Packard color plotter in the mathematics laboratory of the California State University at Northridge. The professor might enjoy Lucy's Lissajous figures. Achkinazi has a program that draws straight lines between corresponding points on a pair of such figures. In this way Lucy produces wild curves clad in a kind of moiré sheen.

Tom Dorn of Vancouver, British Columbia, recommends his own program BUMBLEBEE. It incorporates the following parametric equations in which the constant a can be varied:

$$x = 2\sin(at)$$
$$y = e^{t\sin t}.$$

Temple H. Fay of the University of Southern Mississippi finds polar curves, which are plotted in terms of coordinates (r,θ) rather than (x,y), useful in teaching calculus. The professor plots a butterfly with the aid of sine, cosine and exponential functions:

$$r = e^{\cos\theta} - 2\cos(4\theta) + \sin^5(\theta/12).$$

Commercial and quasi-commercial interest runs rampant in the area of curves. There are products aplenty to aid the amateur charter of curvilinear complexity. For example, David E.-B. Kennedy, a mathematics teacher at the Langley Secondary School in Langley, British Columbia, is enthusiastic about the Casio fx-7000G calculator-plotter. This hand-held marvel displays miniature stepped plots of virtually any function on a 1.5-by-2-inch display.

SPIA, an apparently comprehensive mathematics program, allows users to construct and plot formulas of almost any type. Moreover, it includes special manipulations such as Fourier transforms for those who want to understand signal processing. Interested readers can write to Moonshadow Software, P.O. Box 5974, Baltimore, Md. 21208.

Finally, I have heard from a shadowy organization called MAL (an acronym for Maths Algorithm Library) at P.O. Box 531, Wynnum, Brisbane, Q 4178, Australia. An amusing flyer promises MALtreatment to readers interested in MALfunctions. MALpractice is easy, according to MALadministrator Dr. P. ffyske Howden.

FURTHER READING

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